

ASYMPTOTIC SPEED OF PROPAGATION FOR A VISCOUS SEMILINEAR PARABOLIC EQUATION

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ABSTRACT. We characterize the asymptotic speed of propagation of almost planar solutions to a semilinear viscous parabolic equation, with periodic nonlinearity.

INTRODUCTION

We are interested in the asymptotic behavior, as $\varepsilon \rightarrow 0$, of solutions to the following problem

$$(1) \quad u_t^\varepsilon - \varepsilon^\alpha \Delta u^\varepsilon + g\left(\frac{u^\varepsilon}{\varepsilon}\right) = 0 \quad x \in \mathbb{R}^n, t > 0$$

where $\alpha \in [0, 1)$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz continuous, 1-periodic function, with $g(v) \leq 0$.

If we perform the parabolic rescaling

$$v^\varepsilon(x, t) = \frac{1}{\varepsilon} u^\varepsilon(\varepsilon x, \varepsilon^{2-\alpha} t),$$

equation (1) becomes

$$(2) \quad v_t^\varepsilon - \Delta v^\varepsilon + \varepsilon^{1-\alpha} g(v^\varepsilon) = 0.$$

Therefore, the analysis of the limit as $\varepsilon \rightarrow 0$ of solutions to (1) is related to the long time behavior of solutions of (2).

When $\varepsilon = 1$ or equivalently $\alpha = 1$, the long time behavior has been considered in the literature by several authors, also in the case the laplacian is substituted by a more general elliptic operator, and the existence of special solutions such as traveling or pulsating waves has been established (see [9, 3, 14, 10, 2, 13, 4, 5, 7] and references therein). When $\int_0^1 g = 0$, in [12, 1] it has been studied the limit behavior of solutions of the following long time rescaling of (1)

$$u_t^\varepsilon - \Delta u^\varepsilon + \frac{1}{\varepsilon} g\left(\frac{u^\varepsilon}{\varepsilon}\right) = 0 \quad x \in \mathbb{R}^n, t > 0.$$

In this note we are interested in the case $\alpha \in [0, 1)$. The main result is Theorem 3.1 which provides the effective speed of propagation of solutions to (1) starting from almost planar initial datum. In particular we show that the solution to (1) with initial datum $u^\varepsilon(x, 0) = p \cdot x + k$ satisfies

$$\lim_{\varepsilon \rightarrow 0} u^\varepsilon(x, t) = p \cdot x + k + c(p)t \quad \text{uniformly in } x, t$$

where the average positive speed $c(p)$ depends on g .

The proof of this result relies on maximum principle arguments and on a priori bounds on the gradient to solutions to (1) with planar initial data. These bounds are provided by Theorem 1.1 and are based on the so called Bernstein method.

The interesting feature we observe is that in the case $\alpha < 1$, the asymptotic speed of propagation $c(p)$ is just lower semicontinuous, but not continuous with respect to p ,

whereas when $\alpha = 1$, the speed $c(p)$ depends continuously on p , as it has been proved in [9, 5]. In particular, we show in Proposition 2.1 that

$$c(p) = \begin{cases} -\int_0^1 g(s)ds & p \neq 0 \\ -(\int_0^1 g^{-1}(s)ds)^{-1} & p = 0 \end{cases}$$

so that the limit speed is *discontinuous* with respect to the slope p . Such phenomenon is unusual in homogenization problems, and indicates that the effective limit of (1) as $\varepsilon \rightarrow 0$ is governed by a differential operator which is discontinuous in the gradient entry. An equation similar to (1), with $n = 1$ and $\alpha = 0$, giving rise to a discontinuous effective limit problem, has been considered in [8].

This makes the analysis of this limit more challenging, and will be the topic of a paper in preparation [6], where we will consider the long time behaviour and the asymptotic limit of solutions to

$$(3) \quad u_t^\varepsilon - \varepsilon^\alpha \Delta u^\varepsilon + g\left(\frac{x}{\varepsilon}, \frac{u^\varepsilon}{\varepsilon}\right) = 0 \quad x \in \mathbb{R}^n, t > 0$$

where $g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous and \mathbb{Z}^{n+1} -periodic.

Acknowledgments. The authors would like to thank the mathematics departments of the Universities of Padova, Firenze and Pisa for the kind hospitality during the preparation of this work. A.C. was partially supported by the GNAMPA Project 2015 “Processi di diffusione degeneri o singolari legati al controllo di dinamiche stocastiche”. N.D was partially supported by the University of Padova through the Visiting Scientist Programme, and by the Leverhulme Trust through RPG-2013-261. M.N. was partially supported by the University of Pisa via Grant PRA-2015-0017.

1. A PRIORI ESTIMATES ON THE GRADIENT

We consider in this section the more general case in which $g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz, \mathbb{Z}^{n+1} periodic function.

We introduce, for any $p \in \mathbb{R}^n$, the initial value problem

$$(4) \quad \begin{cases} v_t^\varepsilon - \Delta v^\varepsilon + \varepsilon^{1-\alpha} g(x, v^\varepsilon) = 0 \\ v^\varepsilon(x, 0) = p \cdot x. \end{cases}$$

We recall that this problem admits a unique solution $v^\varepsilon \in C^{2+\gamma, 1+\gamma/2}$ for all $\gamma \in (0, 1)$. Moreover we recall also that, due to Lipschitz regularity of g , a standard comparison principle among sub and supersolutions to (4) holds (see [15]).

We provide an apriori bound on the oscillation and on the gradient of the solutions to (4), which will be useful in the following. The approach is based on the so called Bernstein type method.

In the following we let $w^\varepsilon(x, t) = v^\varepsilon(x, t) - p \cdot x$, where v^ε is the unique solution to (4). Notice that w^ε solves

$$(5) \quad \begin{cases} w_t - \Delta w + \varepsilon^{1-\alpha} g(x, w + p \cdot x) = 0 \\ w(x, 0) = 0. \end{cases}$$

Theorem 1.1. *Given $p \in \mathbb{R}^n$, then there exists $\varepsilon_0 = \varepsilon(|p|)$ such that for every $\varepsilon \leq \varepsilon_0$ and every $t \geq 0$ there holds*

$$\sup_{x \in \mathbb{R}^n} |Dw^\varepsilon(x, t)| \leq C_n \varepsilon^{\frac{1-\alpha}{4}} (1 + |p|) \|g\|_{1, \infty}$$

and

$$\sup_{x \in \mathbb{R}^n} w^\varepsilon(x, t) - \inf_{x \in \mathbb{R}^n} w^\varepsilon(x, t) \leq 2 + C_n \varepsilon^{\frac{1-\alpha}{4}} (1 + |p|) \|g\|_{1,\infty}$$

where C_n is a constant depending on the space dimension and $\|g\|_{1,\infty}$ is the Lipschitz norm of g .

Proof. The proof of the theorem is based on similar arguments as in [9, Thm 2.4, Cor. 2.5]. We divide the proof in several steps.

Step 1: we prove that for every $t \geq 0$

$$(6) \quad \text{osc}(w^\varepsilon(\cdot, t), [0, 1]^n) \leq \text{osc}(w^\varepsilon(\cdot, t), \mathbb{R}^n) \leq \text{osc}(w^\varepsilon(\cdot, t), [0, 1]^n) + 2,$$

where $\text{osc}(w^\varepsilon(\cdot, t), A) = \sup_{x \in A} w^\varepsilon(x, t) - \inf_{x \in A} w^\varepsilon(x, t)$.

Fix $t > 0$ and $\ell \in \mathbb{Z}^n$ and let $\gamma = [p \cdot \ell] + 1 - p \cdot \ell \in (0, 1]$. We define $\hat{w}^\varepsilon(x, t) := w^\varepsilon(x - \ell, t) + \gamma$. Observe that, due to periodicity of g , it satisfies the same equation as w^ε : $\hat{w}_t^\varepsilon(x, t) = \Delta w^\varepsilon(x - \ell, t) - \varepsilon^{1-\alpha} g(x - \ell, w^\varepsilon(x - \ell, t) + p \cdot (x - \ell)) = \Delta \hat{w}^\varepsilon - \varepsilon^{1-\alpha} g(x, \hat{w}^\varepsilon + p \cdot x)$. Moreover $\hat{w}^\varepsilon(x, 0) := w^\varepsilon(x - \ell, 0) + \gamma = \gamma > 0$. Therefore by the comparison principle, we obtain

$$w^\varepsilon(x, t) \leq w^\varepsilon(x - \ell, t) + \gamma \leq w^\varepsilon(x - \ell, t) + 1 \quad \forall x \in \mathbb{R}^n, \ell \in \mathbb{Z}^n.$$

This implies immediately (6).

Step 2: properties of the function $W^\varepsilon(t) = \sup_{x \in \mathbb{R}^n} w^\varepsilon(x, t)$.

For simplicity from now on we consider the case in which $g \in C^\infty(\mathbb{R}^{n+1})$. The case of g just Lipschitz is recovered by a standard approximation procedure.

Note that, by comparison principle, $0 \leq w^\varepsilon(x, t) \leq \varepsilon^{1-\alpha} \|g\|_\infty t$, for all x, t . Moreover, again by comparison principle,

$$\inf_{y \in \mathbb{R}^n} w^\varepsilon(y, s) \leq w^\varepsilon(x, t) \leq \sup_{y \in \mathbb{R}^n} w^\varepsilon(y, s) + \varepsilon^{1-\alpha} (t - s) \|g\|_\infty \quad \forall t \geq s.$$

This implies that the function

$$W^\varepsilon(t) = \sup_{x \in \mathbb{R}^n} w^\varepsilon(x, t)$$

satisfies, in viscosity sense,

$$(7) \quad W_t^\varepsilon \leq \varepsilon^{1-\alpha} \|g\|_\infty.$$

Let $\lambda > 0$ to be fixed later. We consider the function

$$\phi^\varepsilon(x, t) = \frac{|Dw^\varepsilon(x, t)|^2}{2} - \lambda w^\varepsilon(x, t) + \lambda W^\varepsilon(t).$$

Note that $\frac{|Dw^\varepsilon(x, t)|^2}{2} \leq \phi^\varepsilon(x, t) \leq \frac{|Dw^\varepsilon(x, t)|^2}{2} + \lambda \text{osc}(w^\varepsilon(\cdot, t), \mathbb{R}^n)$, and moreover $\phi^\varepsilon(x, 0) = 0$. Let $\Phi^\varepsilon(t) = \sup_{x \in \mathbb{R}^n} \phi^\varepsilon(x, t)$.

Step 3: we prove that choosing $\lambda = C_\varepsilon^{(1-\alpha)/2} n^{1/2} (1 + |p|) (\|Dg\|_\infty + \|g\|_\infty)$, for $C > 1$, then for every ε sufficiently small

$$\Phi^\varepsilon(t) \leq \lambda \sup_{t \in [0, T]} \text{osc}(w^\varepsilon(\cdot, t), \mathbb{R}^n)$$

for every $t \in [0, T]$, where $T > 0$ is the stopping time

$$T = \inf\{s \geq 0 : \sup_x |Dw^\varepsilon(x, s)| = 1\}.$$

Let t^* such that $\Phi^\varepsilon(t^*) \geq \Phi^\varepsilon(t)$ for every $t \in [0, T]$. If $t^* = 0$, then $\Phi^\varepsilon(t) \leq 0$ for every t and we are done. Assume that $t^* > 0$. It is easy to show that there exists a sequence $x_k(t^*)$ such that

$$\phi^\varepsilon(x_k(t^*), t^*) \rightarrow \Phi^\varepsilon(t^*).$$

We can choose the sequence such that

$$D\phi^\varepsilon(x_k(t^*), t^*) \rightarrow 0 \quad \lim_k \Delta\phi^\varepsilon(x_k(t^*), t^*) \leq 0.$$

Finally,

$$\lim_k \phi_t^\varepsilon(x_k(t^*), t^*) \geq 0.$$

We claim that, choosing appropriately λ , $\lim_k |Dw^\varepsilon(x_k(t^*), t^*)| = 0$. If it is true, then

$$\Phi^\varepsilon(t) \leq \Phi^\varepsilon(t^*) \leq \lambda \operatorname{osc}(w^\varepsilon(\cdot, t^*), \mathbb{R}^n) \leq \lambda \sup_{t \in [0, T]} \operatorname{osc}(w^\varepsilon(\cdot, t), \mathbb{R}^n),$$

and then we get the desired conclusion.

Assume by contradiction that the claim is not true. Then, for every λ , there exists a subsequence such that $\lim_k |Dw^\varepsilon(x_k(t^*), t^*)| > 0$. We compute

$$(8) \quad \begin{cases} \phi_{x_i}^\varepsilon = \sum_j w_{x_i, x_j}^\varepsilon w_{x_j}^\varepsilon - \lambda w_{x_i}^\varepsilon \\ \phi_t^\varepsilon = \sum_j w_{x_j, t}^\varepsilon w_{x_j}^\varepsilon - \lambda w_t^\varepsilon + \lambda W_t^\varepsilon \\ \phi_{x_i, x_i}^\varepsilon = \sum_j w_{x_i, x_i, x_j}^\varepsilon w_{x_j}^\varepsilon + \sum_j (w_{x_i, x_j}^\varepsilon)^2 - \lambda w_{x_i, x_i}^\varepsilon. \end{cases}$$

We recall that w^ε solves

$$(9) \quad w_t^\varepsilon - \Delta w^\varepsilon + \varepsilon^{1-\alpha} g(x, w^\varepsilon + p \cdot x) = 0.$$

Differentiating the previous equation with respect to x_j we get

$$w_{x_j, t}^\varepsilon = \sum_i w_{x_i, x_i, x_j}^\varepsilon - \varepsilon^{1-\alpha} g_{x_j}(x, w^\varepsilon + p \cdot x) - \varepsilon^{1-\alpha} g_v(x, w^\varepsilon + p \cdot x) (w_{x_j}^\varepsilon + p_j).$$

So, using this equation, (8), (9) and (7), we get

$$\begin{aligned} \phi_t^\varepsilon &= \sum_j \sum_i w_{x_i, x_i, x_j}^\varepsilon w_{x_j}^\varepsilon - \varepsilon^{1-\alpha} \sum_j w_{x_j}^\varepsilon (g_{x_j} + p_j g_v) - \varepsilon^{1-\alpha} \sum_j g_v(w_{x_j}^\varepsilon)^2 - \lambda w_t^\varepsilon + \lambda W_t^\varepsilon \\ &= \Delta\phi^\varepsilon - \sum_i \sum_j (w_{x_i, x_j}^\varepsilon)^2 + \lambda \Delta w^\varepsilon - \varepsilon^{1-\alpha} \sum_j w_{x_j}^\varepsilon (g_{x_j} + p_j g_v) - \varepsilon^{1-\alpha} \sum_j g_v(w_{x_j}^\varepsilon)^2 - \lambda w_t^\varepsilon + \lambda W_t^\varepsilon \\ &= \Delta\phi^\varepsilon - \sum_i \sum_j (w_{x_i, x_j}^\varepsilon)^2 + \varepsilon^{1-\alpha} [\lambda g - \sum_j w_{x_j}^\varepsilon (g_{x_j} + p_j g_v) - \sum_j g_v(w_{x_j}^\varepsilon)^2] + \lambda W_t^\varepsilon \\ &\leq \Delta\phi^\varepsilon - \sum_i \sum_j (w_{x_i, x_j}^\varepsilon)^2 + 2\varepsilon^{1-\alpha} \lambda \|g\|_\infty + \varepsilon^{1-\alpha} \|Dg\|_\infty (1 + |p|) \left(\sum_j |w_{x_j}^\varepsilon| + \sum_j |w_{x_j}^\varepsilon|^2 \right) \\ &\leq \Delta\phi^\varepsilon - \sum_i \sum_j (w_{x_i, x_j}^\varepsilon)^2 + 2\varepsilon^{1-\alpha} \lambda \|g\|_\infty + \sqrt{n} \varepsilon^{1-\alpha} \|Dg\|_\infty (1 + |p|) (|Dw^\varepsilon| + |Dw^\varepsilon|^2). \end{aligned}$$

We multiply the first equation in (8) by $w_{x_i}^\varepsilon$ and compute at $x_k(t^*)$: we have that

$$\lim_k \sum_i \sum_j w_{x_i, x_j}^\varepsilon(x_k(t^*)) w_{x_j}^\varepsilon(x_k(t^*)) w_{x_i}^\varepsilon(x_k(t^*)) - \lambda |Dw^\varepsilon(x_k(t^*))|^2 = 0.$$

By Cauchy-Schwartz inequality

$$\sum_i \sum_j w_{x_i, x_j}^\varepsilon w_{x_j}^\varepsilon w_{x_i}^\varepsilon \leq |Dw^\varepsilon| \sum_i |w_{x_i}^\varepsilon| \left[\sum_j (w_{x_i, x_j}^\varepsilon)^2 \right]^{1/2} \leq |Dw^\varepsilon|^2 \left(\sum_i \sum_j (w_{x_i, x_j}^\varepsilon)^2 \right)^{\frac{1}{2}}.$$

This implies, since $|Dw^\varepsilon(x_k(t^*))| \neq 0$, that

$$\lim_k \sum_i \sum_j (w_{x_i, x_j}^\varepsilon(x_k(t^*)))^2 \geq \lambda^2.$$

We compute (10) at $x_k(t^*)$, recalling the definition of T , of the sequence $x_k(t^*)$ and (10) we get

$$\begin{aligned}
 0 &\leq \lim_k [\phi_t^\varepsilon(x_k(t^*), t^*) - \Delta \phi^\varepsilon(x_k(t^*), t^*)] \\
 &\leq -\lambda^2 + 2\varepsilon^{1-\alpha} \lambda \|g\|_\infty + \sqrt{n} \varepsilon^{1-\alpha} \|Dg\|_\infty (1 + |p|) \sup_{t \in [0, T]} \sup_x (|Dw^\varepsilon|^2 + |Dw^\varepsilon|) \\
 (11) \quad &\leq -\lambda^2 + 2\varepsilon^{1-\alpha} \lambda \|g\|_\infty + \sqrt{n} \varepsilon^{1-\alpha} \|Dg\|_\infty (1 + |p|).
 \end{aligned}$$

We claim it is possible to choose λ so that the right hand side of (11) is strictly negative getting then a contradiction. Indeed if we choose $\lambda = C\varepsilon^{(1-\alpha)/2} n^{1/2} (1 + |p|) (\|Dg\|_\infty + \|g\|_\infty)$, with $C > 1$, the claim is true for every ε sufficiently small.

Step 4: conclusion.

By the first step, the definition of Φ^ε and (6), we get that

$$\begin{aligned}
 \sup_x \frac{|Dw^\varepsilon(x, t)|^2}{2} \leq \Phi^\varepsilon(t) &\leq \lambda \sup_{t \in [0, T]} \text{osc}(w^\varepsilon(\cdot, t), \mathbb{R}^n) \\
 &\leq \lambda \sup_{t \in [0, T]} (\text{osc}(w^\varepsilon(\cdot, t), [0, 1]^n) + 2) \\
 &\leq \lambda \sup_{t \in [0, T]} \sup_x |Dw^\varepsilon(x, t)| + 2\lambda \\
 &\leq \sup_{t \in [0, T]} \sup_x \frac{|Dw^\varepsilon(x, t)|^2}{4} + \lambda^2 + 2\lambda,
 \end{aligned}$$

which in turn gives

$$(12) \quad \sup_x |Dw^\varepsilon(t, x)|^2 \leq 4\lambda + 2\lambda^2 \quad \forall t \in [0, T].$$

Recall by the first step that $\lambda = C\varepsilon^{(1-\alpha)/2} n^{1/2} (1 + |p|) (\|Dg\|_\infty + \|g\|_\infty)$, so choosing ε sufficiently small, we obtain $4\lambda + 2\lambda^2 < 1$. This implies, recalling (12), that the stopping time $T = +\infty$.

Finally the desired estimate on the gradient is obtained by (12), recalling the explicit formula for λ , and the estimate on the oscillation is deduced by recalling (6). \square

2. EXISTENCE OF ALMOST-PLANAR SOLUTIONS

In this section, we show in fact that there exists for every $p \in \mathbb{R}^n$ a positive constant $c_\varepsilon(p)$ such that solutions to (1) starting from hyperplanes $z = p \cdot x$ remain at a small distance from hyperplanes with the same normal and moving with (uniformly bounded in ε) speed $c_\varepsilon(p)\varepsilon^{\alpha-1}$. Moreover we study the limit as $\varepsilon \rightarrow 0$ to $c_\varepsilon(p)\varepsilon^{\alpha-1}$: this will be the average speed of hyperplanes.

Theorem 2.1. *Let v^ε be the solution to (4). Then there exists a unique $c_\varepsilon(p) > 0$ such that*

$$|v^\varepsilon(x, t) - p \cdot x - c_\varepsilon(p)t| \leq K$$

where K depends on the Lipschitz norm of g . Moreover

$$0 < c_\varepsilon(p) \leq \varepsilon^{1-\alpha} \|g\|_\infty.$$

The proof of this theorem is based on the apriori estimates provided in Theorem 1.1, and follows as in [9, Thm 3.1] (see also [6] for a proof of a more general result of this kind).

Remark 1. Observe that if u^ε is the solution to (1) with initial datum $u^\varepsilon(x, 0) = p \cdot x$, then $v^\varepsilon(x, t) = \frac{1}{\varepsilon} u^\varepsilon(\varepsilon x, \varepsilon^{2-\alpha} t)$ is the solution to (4). So Theorem 2.1 implies

$$|u^\varepsilon(x, t) - p \cdot x - c^\varepsilon(p) \varepsilon^{\alpha-1} t| \leq K\varepsilon.$$

Remark 2. Note that, when $\alpha = 1$, the speed $c^\varepsilon(p)$ does not depend on ε . When $\alpha = 1$ and the laplacian in (1) is substituted by the mean curvature operator, the existence of traveling wave solutions has been established in [9], under suitable assumptions on the forcing term g .

Proposition 2.1. *Let $c^\varepsilon(p)$ as in Theorem 2.1. Then*

$$(13) \quad \lim_{\varepsilon \rightarrow 0} c^\varepsilon(p) \varepsilon^{\alpha-1} = c(p) = \begin{cases} -\int_0^1 g(s) ds & p \neq 0 \\ -\left(\int_0^1 \frac{1}{g(s)} ds\right)^{-1} & p = 0, \ g < 0 \\ 0 & p = 0, \ \max g = 0. \end{cases}$$

Proof. For $p = 0$, equation (4) becomes

$$\begin{cases} v'(t) + \varepsilon^{1-\alpha} g(v) = 0 \\ v(0) = 0. \end{cases}$$

So, if $g < 0$, by direct integration we get

$$\int_0^{v(t)} \frac{ds}{g(s)} = -\varepsilon^{1-\alpha} t.$$

In particular v is monotone increasing, unbounded and

$$\lim_{t \rightarrow +\infty} \frac{v(t)}{t} = -\varepsilon^{1-\alpha} \left(\int_0^1 \frac{1}{g(s)} ds \right)^{-1} = c^\varepsilon(0).$$

If, on the other hand there exists $v_0 \in [0, 1]$ such that $g(v_0) = 0$, then by comparison $v_0 - 1 \leq v(t) \leq v_0$ for every t , and then $c^\varepsilon(0) = 0$.

For $p \neq 0$, we define $\chi_p : \mathbb{R} \times (0, +\infty) \rightarrow \mathbb{R}$ as $\chi_p(p \cdot x, t) = w^\varepsilon(x, t)$, where w^ε solves (5). Then χ_p solves

$$(14) \quad \begin{cases} (\chi_p)_t - |p|^2 (\chi_p)_{zz} + \varepsilon^{1-\alpha} g(\chi_p(z, t) + z) = 0 & t > 0, z \in \mathbb{R} \\ \chi_p(z, 0) = 0 \end{cases}$$

and is 1-periodic. We integrate in space and time (14) and, using periodicity of χ_p , we obtain

$$\int_0^1 \frac{1}{t} \int_0^t (\chi_p)_t dt dz = -\varepsilon^{1-\alpha} \frac{1}{t} \int_0^t \int_0^1 g(\chi_p + z) dz dt.$$

So

$$\frac{1}{t} \int_0^t (\chi_p)_t dt = \frac{\chi_p(z, t)}{t} \rightarrow c^\varepsilon(p) \quad t \rightarrow +\infty.$$

Using the a priori bounds on the derivative $(\chi_p)_z$ given by Theorem 1.1 and the fact that g is Lipschitz, we get

$$\int_0^1 g(z + \chi(z, t)) dz = \int_0^1 g \left(z + \int_0^1 \chi(\zeta, t) d\zeta \right) dz + O(\varepsilon^{\frac{1-\alpha}{4}}) = \int_0^1 g(z) dz + O(\varepsilon^{\frac{1-\alpha}{4}})$$

where the error term $O(\varepsilon^{\frac{1-\alpha}{4}})$ is independent of t . This gives

$$c^\varepsilon(p) = -\varepsilon^{1-\alpha} \int_0^1 g(z) dz + o(\varepsilon^{1-\alpha}).$$

□

3. ASYMPTOTIC SPEED OF PROPAGATION

In this section we provide the asymptotic speed of propagation of solutions to (1) as $\varepsilon \rightarrow 0$ starting from almost planar initial datum. When the initial datum is an hyperplane, we obtain the homogenization limit of solutions to (4).

Theorem 3.1. *Let u_ε be the solution to*

$$(15) \quad \begin{cases} \frac{\partial u^\varepsilon}{\partial t} - \varepsilon^\alpha \Delta u^\varepsilon + g\left(\frac{u^\varepsilon}{\varepsilon}\right) = 0 & t > 0, x \in \mathbb{R}^n \\ u^\varepsilon(x, 0) = u_0(x). \end{cases}$$

Assume that there exists $p \in \mathbb{R}^n$ such that

$$(16) \quad u_0(x) - p \cdot x = v_0(x) \in L^\infty(\mathbb{R}^n).$$

Then

$$p \cdot x + c(p)t + \inf v_0 \leq \liminf_{\varepsilon \rightarrow 0} u^\varepsilon(x, t) \leq \limsup_{\varepsilon \rightarrow 0} u^\varepsilon(x, t) \leq p \cdot x + c(p)t + \sup v_0$$

where $c(p)$ is defined in (13).

In particular, if $u_0(x) - p \cdot x \equiv k$ for $k \in \mathbb{R}$, then the solution u^ε to (15) satisfies

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t) = p \cdot x + c(p)t + k \quad \text{uniformly in } x \in \mathbb{R}^n, t \geq 0.$$

Proof. We consider the solutions $u_+^\varepsilon, u_-^\varepsilon$ to (15) with initial datum $u_+^\varepsilon(x, 0) = p \cdot x + \varepsilon \left[\frac{\sup v_0}{\varepsilon} \right] + \varepsilon$ and $u_-^\varepsilon(x, 0) = p \cdot x + \varepsilon \left[\frac{\inf v_0}{\varepsilon} \right]$. By comparison $u_-^\varepsilon(x, t) \leq u^\varepsilon(x, t) \leq u_+^\varepsilon(x, t)$ for every x, t . We can rewrite

$$u_-^\varepsilon(x, t) = p \cdot x + \varepsilon w^\varepsilon\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^{2-\alpha}}\right) + \varepsilon \left[\frac{\inf v_0}{\varepsilon} \right]$$

and analogously u_+^ε , where w^ε is the solution to (5).

So we get that

$$(17) \quad u_-^\varepsilon(x, t) = p \cdot x + \varepsilon^{\alpha-1} c^\varepsilon(p)t + \varepsilon \left(w^\varepsilon\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^{2-\alpha}}\right) - c^\varepsilon(p) \frac{t}{\varepsilon^{2-\alpha}} \right) + \varepsilon \left[\frac{\inf v_0}{\varepsilon} \right]$$

and analogously for u_+^ε . By comparison we get that

$$\begin{aligned} \varepsilon \left(w^\varepsilon\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^{2-\alpha}}\right) - c^\varepsilon(p) \frac{t}{\varepsilon^{2-\alpha}} \right) + \varepsilon \left[\frac{\inf v_0}{\varepsilon} \right] &\leq u^\varepsilon(x, t) - p \cdot x - \varepsilon^{\alpha-1} c^\varepsilon(p)t \\ &\leq \varepsilon \left(w^\varepsilon\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^{2-\alpha}}\right) - c^\varepsilon(p) \frac{t}{\varepsilon^{2-\alpha}} \right) + \varepsilon \left[\frac{\sup v_0}{\varepsilon} \right] + \varepsilon. \end{aligned}$$

Recalling Theorem 2.1, and letting $\varepsilon \rightarrow 0$, we get the thesis.

□

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